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BENDING OF ELASTIC PLATES WITH A PHYSICALLY NONLINEAR INCLUSION

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An infinite elastic isotropic plate with an elliptical, physically nonlinear inclusion loaded at infinity by uniformly distributed moments is considered. Surface loads are absent. The problem of the stress–strain state of the plate is solved in a closed form. It is shown that, for reasonably general stress–strain relations for the inclusion, the bending-moment field (and the corresponding curvatures) in the inclusion is homogeneous.

Key words: *pure bending of an infinite plate, elliptical, physically nonlinear inclusion, homogeneous moment field.*

The problem of the stress–strain state of an elastic plane with an elliptical, physically nonlinear inclusion (EPNI) under plane strain or generalized plane stress conditions loaded at infinity by uniformly distributed stresses was considered in [1, 2]. It was shown that, under the assumption of stable deformation of the inclusion, there exists a unique solution of this problem, such that the stress–strain state in the EPNI is homogeneous. In the present paper, the results of [1, 2] are generalized to the bending case using the well-known analogy between plane problems and plate-bending problems in the theory of elasticity [3, 4]. In particular, single-valued relations between the homogeneous moment fields at infinity and in the EPNI are established.

1. Formulation of the Problem. We consider an infinite isotropic elastic plate of uniform thickness h with an EPNI (of the same thickness h) loaded at infinity by uniformly distributed moments M_{kl}^{∞} $(k, l = 1, 2)$. The coordinate system $Ox_1x_2x_3$ is chosen in such a manner that the equation of the boundary L between the elastic medium S and the inclusion S_* has the form $x_k^2 a_k^{-2} = 1$ $(a_1 \geq a_2)$. Here and below, summation from 1 to 2 is performed over repeated indices.

For the plate, the strain–displacement relations are [5]

$$\begin{gathered}\varepsilon_{kl}(x_1, x_2, x_3) = x_3 \varkappa_{kl}, \qquad \varkappa_{kl} = -w_{,kl}, \\ w = w(x_1, x_2), \qquad (x_1 x_2) \in S \cup S_*, \qquad |x_3| \leq h/2,\end{gathered} \tag{1.1}$$

where the subscript after the comma denotes differentiation with respect to the corresponding coordinate. In the absence of surface loads, the equations of equilibrium of the plate are written as [5]

$$Q_k = M_{kl,l}, \qquad Q_{k,k} = 0, \qquad Q_k = \int\limits_{-h/2}^{h/2} \sigma_{3k}\, dx_3, \qquad M_{kl} = \int\limits_{-h/2}^{h/2} \sigma_{kl} x_3\, dx_3, \tag{1.2}$$

where w is the deflection, Q_k and M_{kl} are the transverse shear forces and bending moments, respectively, and σ_{kl} are the stress components. In (1.1) and (1.2), and below, $k, l = 1, 2$.

On the boundary L, the following quantities should be continuous [4]:

$$w, \quad \frac{\partial w}{\partial n}, \quad G, \quad Q + \frac{\partial H}{\partial s}. \tag{1.3}$$

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Here $G = M_{kl}n_kn_l$, $Q = Q_kn_k$, $H = M_{kl}n_kt_l$, n_k and t_k are the components of the unit normal and tangent vectors to the contour L, respectively, and s is the arc length of the contour.

In the elastic domain S, the relations between the moments M_{kl} and curvatures $\varkappa_{kl}$ are given by [4]

$$M_{kl} = D[(1-\nu)\varkappa_{kl} + \nu\varkappa_{nn}\delta_{kl}], \qquad D = Eh^3/[12(1-\nu^2)], \tag{1.4}$$

where δ_{kl} are the components of the unit plane tensor, D is the cylindrical rigidity, E is Young's modulus, and ν is Poisson's ratio.

For the EPNI, the constitutive equations are written as

$$\sigma_{kl} = F_{kl}(\varepsilon_{mn}), \qquad \varepsilon_{kl} = x_3\varkappa_{kl}, \qquad F_{kl}(-\varepsilon_{mn}) = -F_{kl}(\varepsilon_{mn}), \tag{1.5}$$

where F_{kl} are nonlinear differentiable functions of arguments satisfying the inequality

$$\frac{\partial F_{kl}}{\partial \varepsilon_{mn}}\,\xi_{kl}\xi_{mn} > 0 \qquad (\xi_{kl}\xi_{kl} \neq 0),$$

which is equivalent to the condition of stable deformation of the inclusion S_* [2, 6]

$$\Delta\sigma_{kl}\Delta\varepsilon_{kl} > 0 \qquad \text{for} \quad \Delta\varepsilon_{kl}\Delta\varepsilon_{kl} \neq 0 \tag{1.6}$$

($\Delta\sigma_{kl}$ and $\Delta\varepsilon_{kl}$ are the increments of stresses and strains, respectively).

Inequality (1.6) ensures that the following relations implied by (1.2) and (1.5) are solved uniquely for the curvatures $\varkappa_{kl}$:

$$M_{kl} = 2\int\limits_0^{h/2} F_{kl}(x_3\varkappa_{mn})x_3\,dx_3.$$

Indeed, considering the difference between two possible solutions, we obtain

$$\int\limits_0^{h/2} \Delta F_{kl}x_3\,dx_3 = \int\limits_0^{h/2} a_{klmn}\Delta\varkappa_{mn}x_3^2\,dx_3 = 0.$$

Here $a_{klmn} = (\partial F_{kl}/\partial\varepsilon_{mn})\Big|_{\varepsilon_{mn}=\varepsilon^*_{mn}}$ and $\varepsilon^*_{kl} = \lambda\varepsilon^{(1)}_{kl} + (1-\lambda)\varepsilon^{(2)}_{kl}$, where $0 < \lambda < 1$ [6]. Since $\Delta\varkappa_{kl}$ are independent of x_3, this relation yields

$$\int\limits_0^{h/2} a_{klmn}\Delta\varkappa_{mn}\Delta\varkappa_{kl}x_3^2\,dx_3 = 0,$$

which is possible only if $\Delta\varkappa_{kl} = 0$ $(k, l = 1, 2)$.

If the moment field M_{kl} in S_* is homogeneous, i.e., $M_{kl,i} = 0$, the curvature field $\varkappa_{kl}$ is also homogeneous. This conclusion is inferred from the relations

$$M_{kl,i} = 2\int\limits_0^{h/2} \sigma_{kl,i}x_3\,dx_3 = 2\int\limits_0^{h/2} \frac{\partial F_{kl}}{\partial\varepsilon_{mn}}\,\varkappa_{mk,i}x_3^2\,dx_3, \tag{1.7}$$

which follow from (1.5). Setting the right side in (1.7) equal to zero, multiplying by $\varkappa_{kl,i}$, and performing summation over k and l, we obtain the equality

$$\int\limits_0^{h/2} \frac{\partial F_{kl}}{\partial\varepsilon_{mn}}\,\varkappa_{mn,i}\varkappa_{kl,i}x_3^2\,dx_3 = 0$$

(no summation over i is performed), which holds only for $\varkappa_{kl,i} = 0$ $(i, k, l = 1, 2)$.

Examples of functions (1.5) are functions similar to those considered in [5]:

$$\sigma_{kl} = \frac{2}{3}\,x_3\,\frac{\sigma_i}{\varepsilon_i}\,(\varkappa_{kl} + \varkappa_{nn}\delta_{kl}), \qquad \sigma_i = \sigma_i(\varepsilon_i),$$

$$\sigma_i^2 = \sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2 + 3\sigma_{12}^2, \qquad \varepsilon_i = |x_3|\varkappa_i, \tag{1.8}$$

$$\varkappa_i^2 = 4(\varkappa_{11}^2 + \varkappa_{11}\varkappa_{22} + \varkappa_{22}^2 + \varkappa_{12}^2)/3,$$

where σ_i and ε_i are the stress and strain intensities, respectively.

Condition (1.6) is reduced to the inequality $\sigma_i' > 0$ [6], which is equivalent to the inequality $p > 0$ for the power function $\sigma_i = B_0\varepsilon_i^p$. In this case, from (1.8), we obtain the relations between M_{kl} and $\varkappa_{kl}$

$$M_{kl} = D_0\varkappa_i^{p-1}\,\frac{\varkappa_{kl} + \varkappa_{nn}\delta_{kl}}{2}, \qquad D_0 = \frac{8B_0}{3(p+2)}\Big(\frac{h}{2}\Big)^{p+2},$$

which are easily inverted:

$$\varkappa_{kl} = C_0 M_i^{(1-p)/p}(M_{kl} - M_{nn}\delta_{kl}/3),$$

$$M_i^2 = M_{11}^2 - M_{11}M_{22} + M_{22}^2 + 3M_{12}^2, \qquad C_0 = (3/2)(3D_0/4)^{-1/p}.$$

2. Solution of the Problem. As noted above, a homogeneous stress–strain state occurs in the inclusion in a similar problem of an elastic plane under uniformly distributed stresses at infinity. We show that the moment and curvature fields in the EPNI are also homogeneous for the case of pure bending.

It follows from (1.2) and (1.4) that the deflection w in the elastic domain S satisfies the biharmonic equation and, hence, can be expressed in terms of two functions $\varphi(z)$ and $\chi(z)$ of the complex variable $z = x_1 + ix_2$ [3]:

$$2w = \overline{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}. \tag{2.1}$$

Treating the function w as a function of two independent variables z and $\overline{z}$, we obtain the following relations for the moments M_{kl} in S [3, 4, 7]:

$$M_{11} + M_{22} = -2D(1+\nu)[\Phi(z) + \overline{\Phi(z)}] = -4D(1+\nu)w_{,z\overline{z}},$$

$$M_{22} - M_{11} + 2iM_{12} = 2D(1-\nu)[\overline{z}\Phi'(z) + \Psi(z)] = 4D(1-\nu)w_{,zz}, \tag{2.2}$$

$$\Phi(z) = \varphi'(z), \qquad \Psi(z) = \chi''(z).$$

We assume that the moment field M_{kl} (and, hence, the curvature field $\varkappa_{kl}$) in the EPNI is homogeneous. From the relations $w_{,kl} = -\varkappa_{kl}$, with accuracy to a linear function of x_1 and x_2, which has no effect on the stress–strain state in S_*, we find that $2w = -\varkappa_{kl}x_kx_l$ or, in the variables z and $\overline{z}$,

$$8w(z,\overline{z}) = (\varkappa_{22} - \varkappa_{11} + 2i\varkappa_{12})z^2 + (\varkappa_{22} - \varkappa_{11} - 2i\varkappa_{12})\overline{z}^2 - 2(\varkappa_{11} + \varkappa_{22})z\overline{z}. \tag{2.3}$$

The functions $\Phi(z)$ and $\Psi(z)$ in (2.2) should satisfy the boundary conditions on L [4]

$$\Phi(\tau) - \lambda_k\overline{\Phi(\tau)} - [\overline{\tau}\Phi'(\tau) + \Psi(\tau)]\,\mathrm{e}^{2i\alpha} = f_k \qquad (k = 1, 2),$$

$$\lambda_1 = -1, \qquad \lambda_2 = (3+v)/(1-\nu), \tag{2.4}$$

$$f_1 = \mathrm{e}^{i\alpha}\,\frac{d(w_{,2} + iw_{,1})}{ds} = 2i\,\mathrm{e}^{i\alpha}\,\frac{dw_{,z}}{ds}, \qquad f_2 = \frac{1}{D(1-\nu)}\Big[G - i\Big(H + \int\limits_0^s Q\,ds\Big)\Big],$$

where α is the angle between the normal vector to the contour L at the point τ and the Ox_1 axis. The quantities in the expressions for f_1 and f_2 are assumed to be specified on L as functions of the arc coordinates s.

The relations [3, 7]

$$2\,\frac{\partial}{\partial\overline{z}} = \mathrm{e}^{i\alpha}\Big(\frac{\partial}{\partial n} + i\,\frac{\partial}{\partial s}\Big), \qquad 2\,\frac{\partial}{\partial z} = \mathrm{e}^{-i\alpha}\Big(\frac{\partial}{\partial n} - i\,\frac{\partial}{\partial s}\Big)$$

imply that

$$i\,\frac{\partial}{\partial s} = \mathrm{e}^{-i\alpha}\,\frac{\partial}{\partial\overline{z}} - \mathrm{e}^{i\alpha}\,\frac{\partial}{\partial z}.$$

Therefore, f_1 in (2.4) is the boundary value of the function $2(w_{,z\overline{z}} - \mathrm{e}^{2i\alpha}\,w_{,zz})$. From (2.3), we obtain

$$2f_1 = -[\varkappa_{11} + \varkappa_{22} + (\varkappa_{22} - \varkappa_{11} + 2i\varkappa_{12})\,\mathrm{e}^{2i\alpha}]. \tag{2.5}$$

Approaching the boundary L from the domain S_* and bearing in mind that $n_1 = t_2 = \cos\alpha$ and $n_2 = -t_1 = \sin\alpha$ and $M_{kl,i} = 0$ $(i, k, l = 1, 2)$ by virtue of the assumption, for the quantities G, H, and Q in (1.3), we obtain

$$2G = M_{11} + M_{22} - (M_{22} - M_{11})\cos 2\alpha + 2M_{12}\sin 2\alpha, \quad 2H = (M_{22} - M_{11})\sin 2\alpha + 2M_{12}\cos 2\alpha, \quad Q = 0.$$

It follows that

$$2D(1-\nu)f_2 = M_{11} + M_{22} - (M_{22} - M_{11} + 2iM_{12})\,\mathrm{e}^{2i\alpha}. \tag{2.6}$$

Let the domain S be mapped onto the exterior of the unit circle γ in the complex plane ζ [3]:

$$z = \omega(\zeta) = R_0(\zeta + m\zeta^{-1}), \qquad \zeta = \rho\,\mathrm{e}^{i\theta}, \qquad 2R_0 = a_1 + a_2,$$
$$m = (a_1 - a_2)/(a_1 + a_2) \quad (0 \le m < 1). \tag{2.7}$$

For $\rho = 1$, Eqs. (2.4)–(2.6) yield the boundary conditions for the functions $\Phi_1(\zeta) = \Phi(\omega(\zeta))$ and $\Psi_1(\zeta) = \Psi(\omega(\zeta))$, which determine the stress–strain state for $|\zeta| > 1$ [3]:

$$\Phi_1(\sigma) - \lambda_k\overline{\Phi_1(\sigma)} - [\overline{\omega(\sigma)}\Phi_1'(\sigma)/\omega'(\sigma) + \Psi_1(\sigma)]\,\mathrm{e}^{2i\alpha} = \overline{F_k(\sigma)} \quad \text{on} \quad \gamma,$$
$$\overline{F_k(\sigma)} = f_k \quad (k = 1, 2), \qquad \sigma = \mathrm{e}^{i\theta}, \qquad \mathrm{e}^{2i\alpha} = \sigma^2\omega'(\sigma)/\overline{\omega'(\sigma)}. \tag{2.8}$$

Subtraction of the second equality from the first equality in (2.8) yields

$$\Phi_1(\sigma) = (1-\nu)[F_1(\sigma) - F_2(\sigma)]/4 \quad \text{on} \quad \gamma.$$

Then, relations (2.5)–(2.8) and the well-known formulas [3, 7] can be combined to give

$$\Phi_1(\zeta) = A - (B + iC)(m - \zeta^{-2})/(1 - m\zeta^{-2}),$$

$$\Psi_1(\zeta) = \frac{[A_1 - \Phi_1(\zeta) - \overline{\Phi_1}(\zeta^{-1})](m - \zeta^{-2}) - \Phi_1'(\zeta)(\zeta^{-1} + m\zeta)}{1 - m\zeta^{-2}} - (B_1 - iC_1), \tag{2.9}$$

$$\overline{\Phi_1}(\zeta^{-1}) \equiv \overline{\Phi_1(\bar\zeta^{\,-1})} = A - (B - iC)(m - \zeta^2)/(1 - m\zeta^2), \qquad |\zeta| > 1,$$

$$8A = 2(1-\nu)A_1 - (M_{11} + M_{22})/D, \qquad 8B = 2(1-\nu)B_1 + (M_{22} - M_{11})/D,$$

$$4C = (1-\nu)C_1 - M_{12}/D, \qquad 2A_1 = -(\varkappa_{11} + \varkappa_{22}),$$

$$2B_1 = -(\varkappa_{22} - \varkappa_{11}), \qquad C_1 = \varkappa_{12}.$$

Letting $|\zeta| \to \infty$, from (2.2) and (2.9) we obtain the following formulas relating the stress–strain state in the EPNI to that at infinity:

$$M_{11}^\infty + M_{22}^\infty = 4D(1+\nu)(mB - A),$$

$$M_{22}^\infty - M_{11}^\infty + 2iM_{12}^\infty = 2D(1-\nu)\{m(A_1 - 2A) + (1 + m^2)B - B_1 + i[C_1 - (1 - m^2)C]\}.$$

Solving these relations for the curvatures $\varkappa_{kl}$ in the domain S_*, we obtain

$$D(1-\nu)\varkappa_{11} = a_{11}M_{11} + a_{12}M_{22} + b_{11}M_{11}^\infty + b_{12}M_{22}^\infty,$$

$$D(1-\nu)\varkappa_{22} = a_{12}M_{11} + a_{22}M_{22} + b_{12}M_{11}^\infty + b_{22}M_{22}^\infty,$$

$$D(1-\nu)\varkappa_{12} = aM_{12} + bM_{12}^\infty, \qquad a_{11} = F_1^0(m), \qquad a_{22} = F_1^0(-m), \tag{2.10}$$

$$a_{12} = -(1+\nu)/(3+\nu), \qquad a = -(1-\nu)(1-m^2)/[4 - (1-\nu)(1-m^2)],$$

$$b_{11} = F_2^0(m), \qquad b_{22} = F_2^0(-m), \qquad b_{12} = (1-\nu)/[(3+\nu)(1+\nu)], \qquad b = 1 - a,$$

$$F_1^0(x) = -2(1-x)/[(3+\nu)(1+x)],$$

$$F_2^0(x) = [3\nu + 5 + x(1-\nu)]/[(3+\nu)(1+\nu)(1+x)].$$

Relations (2.10) and (1.5) subject to condition (1.6) can be solved uniquely for the moments M_{kl} in the EPNI. Indeed, given M_{kl}^{∞}, the difference of two possible solutions in (2.10) is

$$D(1-\nu)\Delta M_{kl}\Delta\varkappa_{kl} = a_{11}(\Delta M_{11})^2 + 2a_{12}\Delta M_{11}\Delta M_{22} + a_{22}(\Delta M_{22})^2 + 2a(\Delta M_{12})^2 < 0,$$

since $a_{11} < 0$, $a_{11}a_{22} - a_{12}^2 > 0$, and $a < 0$, whereas it follows from (1.6) that

$$\Delta M_{kl}\Delta\varkappa_{kl} = 2\int\limits_0^{h/2} \Delta\sigma_{kl}\Delta\varepsilon_{kl}x_3\,dx_3 > 0 \qquad \text{for} \quad \Delta\varkappa_{kl}\Delta\varkappa_{kl} \neq 0.$$

Hence, $\Delta\varkappa_{kl} = 0$ and $\Delta M_{kl} = 0$ in S_* $(k, l = 1, 2)$.

It should be noted that the solution constructed above for the stress–strain state in the domain $S \cup S_*$ produced by the moments M_{kl}^{∞} applied at infinity is unique, i.e., the stress–strain state in the EPNI is homogeneous. To prove this statement, we use inequality (1.6) and the following reasoning similar to that given in [3] for a plane elastic problem in an infinite domain. Since M_{kl} are limited as $|z| \to \infty$, the functions $\Phi(z)$ and $\Psi(z)$ in (2.2) are holomorphic at infinity. Thus, discarding the logarithmic terms and terms linear in z and $\overline{z}$, which have no effect on the stress–strain state, from the expression for w (2.1), we obtain, as $|z| \to \infty$,

$$\varphi(z) = a_1 z + a_{-1}z^{-1} + \ldots, \qquad \chi(z) = b_2 z^2 + b_{-1}z^{-1} + \ldots,$$

where

$$a_1 = -(M_{11}^{\infty} + M_{22}^{\infty})/[4D(1+\nu)], \qquad b_2 = (M_{22}^{\infty} - M_{11}^{\infty} + 2iM_{12}^{\infty})/[4D(1-\nu)].$$

Since $\operatorname{Im} a_1$ has no effect on the stress–strain state, it is assumed above that $a_1 = \overline{a_1}$. Using the expressions for the functions $\varphi(z)$ and $\chi(z)$, from (2.1) and (2.2) for the difference between two possible solutions corresponding to the moments M_{kl}^{∞}, we infer that Δw is limited as $|z| \to \infty$ and $\partial\Delta w/\partial n = O(z^{-1})$, $\Delta M_{kl} = O(z^{-2})$, and $\Delta Q + \partial\Delta H/\partial s = O(z^{-3})$.

Instead of the domain S, we consider a finite elastic domain S_r whose external boundary is a circumference L_r of radius r and whose internal boundary is the elliptical contour L. From the virtual work equation [5], which is also valid for the domain $S_r \cup S_*$ by virtue of the condition that the quantities in (1.3) are continuous on L, it follows that

$$\int\limits_{-h/2}^{h/2}\int\limits_{S_r\cup S_*} \Delta\sigma_{kl}\Delta\varepsilon_{kl}\,dS\,dx_3 = I, \qquad I \equiv \int\limits_{L_r}\Big[\Big(\Delta Q + \frac{\partial\Delta H}{\partial s}\Big)\Delta w - \Delta G\,\frac{\partial\Delta w}{\partial n}\Big]\,ds.$$

In the integral over L_r, the integrand function is of the order $|z|^{-3} = r^{-3}$, which implies that $I = O(r^{-2})$. As $r \to \infty$, we obtain

$$\int\limits_{-h/2}^{h/2}\int\limits_{S\cup S_*} \Delta\sigma_{kl}\Delta\varepsilon_{kl}\,dS\,dx_3 = 0.$$

By virtue of (1.4) and (1.6), this relation holds only for $\Delta\varepsilon_{kl} = 0$ and $\Delta\sigma_{kl} = 0$ $(k, l = 1, 2)$ everywhere in the domain $S \cup S_*$.

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